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2001 J. Phys. A: Math. Gen. 34 3811

(http://iopscience.iop.org/0305-4470/34/18/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 3811–3821 www.iop.org/Journals/ja PII: S0305-4470(01)19488-6

Growth and integrability in discrete systems

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Received 24 November 2000, in final form 15 February 2001

Abstract

We introduce a new discrete integrability criterion inspired from the recent findings of Ablowitz and collaborators. This criterion is based on the study of the growth of some characteristic of the solutions of a mapping, using Nevanlinna theory. Since the practical implementation of the latter does not always lead to a clear-cut answer, we complement the growth criterion by the singularity confinement property. This combination turns out to be particularly efficient. Its application allows us to recover the known forms of the discrete Painlevé equations and to show that no new ones may exist within a given parametrization.

PACS numbers: 0230G, 0230, 0545

1. Introduction

The explosive growth of interest in integrable discrete systems over the past decade has led to the derivation of new and interesting, integrable mappings but also to the proposal of several integrability detectors. The principle on which the latter are based, is simple in theory. One must first identify some property which is characteristic of all integrable systems (a task already highly nontrivial). Then one must check whether this property is exclusive to integrable systems (a still more delicate task). In the case of discrete systems the two properties which have served as a guide in the quest for integrability detectors are singularities [1, 2] and growth [3]. The importance of singularities in integrability can be understood in analogy to the continuous case [4]. In this spirit we have proposed the singularity confinement [1] property based on the observation that a singularity spontaneously appearing in an integrable mapping disappeared after some iterations. The idea that growth may play a role in the integrability of discrete systems goes back to Arnold [5] and Veselov [6]. The latter has stated epigrammatically: 'integrability has an essential correlation with the weak growth of certain characteristics'. Viallet and collaborators [7, 8] have elaborated further Arnold's idea of complexity and proposed an efficient discrete integrability detector in the form of what they dubbed algebraic entropy.

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A very interesting direction in the detection of discrete integrability was initiated by Ablowitz and collaborators (AHH) [9]. Their starting point was the formal identity of discrete systems and delay equations [10]. As was shown by Yanagihara [11], such equations possess nontrivial solutions which are meromorphic in the complex variable. Once one adopts the latter viewpoint one can consider discrete equations as defined in the complex plane of the independent variable and deploy all the arsenal of complex analysis. The crucial remark of AHH was that at the continuous limit, when the delay equation goes over to a differential one, the singularities *at finite distance* of the latter can be associated to the behaviour of the solutions of the discrete system at infinity. This led to the introduction of methods for the study of the behaviour near ∞ of the solutions of a given mapping and the natural framework for this is Nevanlinna theory [12]. This theory provides tools for the study of the value distribution of meromorphic functions. In particular it introduces the notion of order. The latter is infinite for very-fast-growing functions, while a finite order indicates a moderate growth. It would be reasonable to surmise that an infinite order is an indication of nonintegrability.

Given a discrete system, it is often difficult to obtain a clear-cut answer concerning the order of its solutions. Usually one starts by imposing as many constraints as one can obtain through the implementation of the Nevanlinna approach and then proceed in a different way. At this point various strategies can be adopted. The one of Ablowitz *et al* consists of requiring that (the series expansions of) the solutions be free from digamma functions. In this paper we shall propose a different approach.

Our method relies on Nevanlinna theory for the estimation of the growth of the solutions of a given discrete system. However, since the order may depend on the precise coefficients of the equation and their dependence on the independent variable, throughout this paper our starting point is autonomous. This assumption greatly simplifies the approach and leads to more precise results. Once the largest possible number of constraints is obtained through this method we proceed to the second step by implementing the singularity confinement criterion. Our assumption is that once the mappings with very-fast-growing solutions are eliminated, singularity confinement is sufficient for integrability. At this second step we are again dealing with autonomous systems. The third and final step consists of deautonomization, again using the singularity confinement criterion. We believe that this three-tiered approach is easier to implement than that of Ablowitz *et al* and has a wider range of applicability.

In this paper, after a brief recall of Nevanlinna theory, we shall illustrate our method by applying it to the family of mappings introduced in [13] (known as QRT mappings) which, through deautonomization, lead to discrete Painlevé equations [14]. We shall recover the already known forms of the discrete Painlevé equations but we shall also show that no new integrable forms may exist besides the already known ones.

2. A brief recall of Nevanlinna theory

As we stated in the introduction, we expect the integrability of a mapping to be conditioned by the behaviour of its solutions when the (complex) independent variable goes to infinity. The main tool for the study of the value distribution of entire and meromorphic functions is the Nevanlinna characteristic (and various quantities related to the latter). The Nevanlinna characteristic of a function f, denoted by $T(r; f)$ measures the 'affinity' of f for the value ∞. It is usually represented as the sum of two terms: the frequency of poles and the contribution from the arcs $|z| = r$ where $|f(z)|$ is large. From the characteristic one can define the order of a meromorphic function: $\sigma = \limsup_{r\to\infty} \log T(r; f)/\log r$. When f is rational, $T(r; f) \propto \log r$ and $\sigma = 0$. A fast-growing function such as e^{e^z} leads to $T \propto e^r$ and thus $\sigma = \infty$.

An explicit expression of the Nevanlinna characteristic can be given in terms of the counting and proximity functions related to the two contributions we mentioned above. We have

$$T(r;\, f) = N(r;\, f) + m(r;\, f) \tag{2.1}$$

where

$$N(r;\, f) = \int_0^r \frac{n(t;\, f) - n(0;\, f)}{t}\, \mathrm{d}t + n(0;\, f)\log r \tag{2.2}$$

is the pole-counting contribution where $n(r;\, f)$ is the number of poles of f, including multiplicities, for $|z| \leqslant r$. The proximity function $m(r;\, f)$ is given by

$$m(r;\, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\mathrm{e}^{\mathrm{i}\theta})|\, \mathrm{d}\theta \tag{2.3}$$

where $\log^+ g = \max(0, \log g)$. We must point out that the affinity of f for ∞, as measured by T, is the same as its affinity for 0 or any finite value a, up to terms which may be of $\mathcal{O}$ $(\log r)$ when r is *small*, but which, when r is sufficiently large (depending on the function f and the value a) remain bounded. In what follows, we shall introduce the symbols $\asymp$, $\preceq$ and $\prec$ which denote equality, inequality and strict inequality respectively *up to a function of r which remains bounded when* $r \to \infty$. The two basic relations which reproduce the statement on the affinity of f for ∞, 0 or a are

$$T(r;\, 1/f) \asymp T(r;\, f) \tag{2.4}$$

$$T(r;\, f - a) \asymp T(r;\, f). \tag{2.5}$$

Using those two identities, we can easily prove that the characteristic function of a homographic transformation of f (with constant coefficients) is equal to $T(r;\, f)$ up to a bounded quantity. It is straightforward to prove that

$$T(r;\, f^n) \asymp |n| T(r;\, f) \tag{2.6}$$

and from a theorem due to Valiron [15] we have

$$T\left(r;\, \frac{P(f)}{Q(f)}\right) \asymp \sup(p, q) T(r;\, f) \tag{2.7}$$

where P and Q are polynomials in f with constant coefficients, of degrees p and q respectively, provided the rational expression P/Q is irreducible.

Let us give also some useful classical inequalities

$$T(r;\, fg) \preceq T(r;\, f) + T(r;\, g) \tag{2.8}$$

$$T(r;\, f + g) \preceq T(r;\, f) + T(r;\, g). \tag{2.9}$$

Another inequality, which was first given in [16], is

$$T(r;\, fg + gh + hf) \preceq T(r;\, f) + T(r;\, g) + T(r;\, h). \tag{2.10}$$

The proof of this inequality is easy, using the fact that $T(r;\, f)$ is the sum of the pole counting and the proximity contributions. For the former we remark that the density of poles of $fg + gh + hf$ cannot be higher than the sum of those of f, g and h. As a consequence, for the part under the integral in (2.2) we have the desired inequality. The $n(0, f)\log r$ term, for $r < 1$, may introduce a contribution going against the inequality we wish to prove, but whenever $r > 1$, the contribution is in the right direction and we thus have indeed $N(r;\, fg + gh + hf) \preceq N(r;\, f) + N(r;\, g) + N(r;\, h)$. For the proximity function, we start by remarking that $|fg + gh + hf| < 3\sup(|fg|, |gh|, |hf|)$. Taking the $\log^+$ of both members, we can strengthen the inequality by adding to the right-hand side (rhs) the $\log^+$ of the one of the $|f|$, $|g|$, $|h|$ that does not enter in each term of the sup. We have thus: $\log^+|fg + gh + hf| < \log 3 + \log^+|f| + \log^+|g| + \log^+|h|$ which leads to $m(r;\, fg+gh+hf) \preceq m(r;\, f)+m(r;\, g)+m(r;\, h)$. Adding the two inequalities for N and m we find (2.10). It can be generalized to

$$T\left(r;\sum_{J\subseteq I}\alpha_J\left(\prod_{j\in J} f_j\right)\right) \preceq \sum_{i\in I} T(r;\, f_i) \tag{2.11}$$

for constant α_J values.

One last property of the Nevanlinna characteristic was obtained by Ablowitz *et al* [9]. In our notation it reads

$$T(r;\, f(z\pm 1)) \preceq (1+\epsilon)T(r+1;\, f(z)). \tag{2.12}$$

This relation (which is valid for r large enough for any given ϵ) makes it possible to have access to the characteristic, and thus the order, of the solution of some difference equations.

3. A practical integrability criterion for discrete systems

The application of the Nevanlinna-based approach will be used here in order to examine the integrability of a selected family of mappings and at the same time propose a novel integrability criterion. The first necessary condition for a noninfinite order, obtained when evaluating Nevanlinna characteristics, is usually insufficient for the order to be finite. Here we will supplement these estimations by the application of the singularity confinement criterion. The resulting combination, as we shall show, is a powerful heuristic tool for the detection of integrability in discrete systems.

The first step, given a mapping, is to use Nevanlinna characteristic techniques in order to estimate the growth rate of the solutions. Since for nonautonomous equations this rate depends on the growth rate of the coefficients of the equation we opt for a simple approach: at this first step we consider *only autonomous* mappings, i.e. mappings the coefficients of which are constants.

The discrete equations we shall examine here are three-point mappings of the general form

$$A(x_n, x_{n-1}, x_{n+1}) = B(x_n) \tag{3.1}$$

where, in principle, A is polynomial and B is rational. Moreover, in this case we shall consider A to be linear separately in $x_{n\pm1}$. Following the approach of AHH we consider equation (3.1) as a delay equation in the complex domain and evaluate the Nevanlinna characteristic of both members of the equality using (2.12) and (2.7). We find

$$u(1+\epsilon)T(r+1;\, x) + vT(r;\, x) \succeq wT(r;\, x) \tag{3.2}$$

(with $u = 2$ if A is linear in $x_{n\pm1}$) for appropriate values of v and w. From (3.2) we have

$$T(r+1;\, x) \succeq \frac{w-v}{u(1+\epsilon)}T(r;\, x). \tag{3.3}$$

Now if $w > u + v$, for large enough values of r one can always choose ϵ to be small enough so that $\lambda \equiv w - v/u(1+\epsilon)$ becomes strictly greater than unity. The precise meaning of (3.3) is that for large enough values of r we have

$$T(r+1;\, x) \geqslant \lambda T(r;\, x) - C \tag{3.4}$$

for some C independent of r. The case where C is negative is trivial: $T(r+k;\, x) \geqslant \lambda^k T(r;\, x)$. For positive values of C we have

$$T(r+1;\, x) - \frac{C}{\lambda-1} \geqslant \lambda\left(T(r;\, x) - \frac{C}{\lambda-1}\right). \tag{3.5}$$

Thus, whenever $T(r; x)$ is an unbounded-growing function of r (i.e. $T \succ 0$), then where r is large enough the rhs of this inequality becomes strictly positive and iterating (3.5) we see that $T(r + k; x)$ diverges at least as fast as λ^k, thus $\log T(r; x) > r \log \lambda$ and the order σ of x is infinite. Thus, according to the AHH hypothesis the mapping cannot be integrable. The only way out is if $T(r; x)$ is a constant which means that x is itself a constant, since the slowest possible growth of the Nevanlinna characteristic for a nonconstant meromorphic function is $T(r; f) \asymp \log r$, for f a homographic function of z. Giving that the mapping is rational, there can only be a finite number of constant solutions. We could in principle have had an infinite number of constant solutions if the identity $A(x_n, x_n, x_n) \equiv B(x_n)$ were true. However this would imply $w \leqslant u + v$. Thus when $w > u + v$ in (3.2) the only possible finite-order solutions are (a finite number of) constant solutions, *all* the remaining ones having $\sigma = \infty$.

The advantage of working with autonomous mappings lies in the fact that we can precisely control the corrective terms in the inequalities for T. Had we worked with nonautonomous systems, we would have had unbounded corrective terms. For instance if the coefficients depend rationally on z, there would be corrective terms of order $\mathcal{O}(\log r)$ and we would have been unable to exclude (finite-order) rational solutions. Though one could suspect that the generic solution is not rational, one could not easily prove this fact in the nonautonomous case. However, in our approach we consider nonautonomous equations as obtained from autonomous ones through a deautonomization procedure. (By deautonomization we mean that we allow the coefficients of the mapping to be functions of the independent discrete variable.) This procedure will never transform a $\sigma = \infty$ solution into a finite σ one. So the generic solution will have $\sigma = \infty$ whenever $w > u + v$ in (3.2) even in the nonautonomous case. The rational solutions that we cannot exclude can only come, through the deautonomization procedure, from the finite- (in effect, zero-)order constant solutions, of which there is a finite number.

As we just saw, this first step puts severe constraints on the discrete equations at hand. However, usually, these constraints are not restrictive enough so as to fix completely the form of the mapping, hence the necessity of the second step. (At this point our approach diverges from that of AHH.) Once the constraints of the first step are implemented, we pursue, using singularity confinement, in order to constrain further our discrete equation. The application of the singularity confinement criterion is based on the assumption that for an *integrable* mapping all singularities spontaneously appearing at some step do disappear after some iterations. Thus all autonomous equations that do not satisfy confinement are rejected at this second step.

The third step consists of the deautonomization of the system using once again the confinement criterion. We thus obtain a mapping which (hopefully) satisfies the Nevanlinna criterion for low growth of the solutions as well as the singularity confinement. The major difficulty lies in the fact that the practical evaluation of the Nevanlinna characteristic gives a clear-cut answer as to mappings the solutions of which must be (generically) of infinite order, but this does not mean that all the remaining ones have their generic solution of finite order. Particular care is needed in the application of this criterion, lest one proclaims of finite-order systems which have in fact infinite-order solutions.

In what follows, we are going to examine systems belonging to the family of discrete Painlevé equations (d-$\mathbb{P}$s) [17]. Our starting point will be functional forms related to the various members of the family of 'standard' d-$\mathbb{P}$s.

Following the three-tiered approach we sketched earlier, we start with autonomous systems. Since in every case examined, the application of the Nevanlinna criterion combined to that of singularity confinement leads to precisely the QRT [13] forms, we shall not proceed to the third step, namely deautonomization. As a matter of fact the deautonomization of the QRT mappings, associated with the standard family of d-$\mathbb{P}$s was given in full detail in [18].

We start with the equations of the d-$\mathrm{P}_{\mathrm{I/II}}$ family. These have been examined in detail by

AHH, but we present here the results for completeness sake. They also give us the opportunity to present the differences between our arguments and the ones of AHH. The starting point is the *autonomous* equation

$$x_{n+1} + x_{n-1} = \frac{P(x_n)}{Q(x_n)}. \tag{3.6}$$

Whenever the condition $w > u + v$, obtained in section 2, is satisfied, we know that the generic solution will have infinite order, since only a finite number of constant solutions can have finite (in fact, zero) order. For equation (3.6) $u = 2$, $v = 0$ and w is the maximum of the degrees of P and Q. Thus w can be at most 2 for the order of the generic solution not to be infinite. There is a subtle difference in our reasoning compared to that of AHH. The latter authors conclude that, if P and Q depend rationally on n, if $w > 2$ *and* if the solution is not rational, *then* the order is infinite, but they cannot exclude the possibility that for some choice of P, Q with $w \geqslant 3$ all solutions may be rational. For us, using constant P and Q we can conclude that if $w > 3$ the generic solution has $\sigma = \infty$, without any other assumption on its rationality. The only solutions with finite order are constants and there exists a finite number of them. Then upon deautonomization, the generic $\sigma = \infty$ solution cannot recover a finite order, while some very special zero-order rational solutions may arise from the constant solutions of the autonomous case.

Let us examine all mappings with $w \leqslant 2$. This will help us illustrate what we mean by singularity confinement and deautonomization. We start by rewriting (3.6) in the case of a quadratic numerator and denominator as

$$x_{n+1} + x_{n-1} = -\frac{\eta x_n^2 + \epsilon x_n + \zeta}{\alpha x_n^2 + \beta x_n + \gamma} \tag{3.7}$$

where we assume that α, β are not both zero. In order to implement singularity confinement we distinguish the two cases $\alpha \neq 0$ and $\alpha = 0$. Let us start with the latter and take $\beta = 1$. By translating x we obtain the mapping with $\gamma = 0$ i.e. with just x_n in the denominator. Clearly if, due to the initial conditions chosen, $x_n = 0$ for some n, then $x_{n+1} = \infty$. This singularity will propagate indefinitely unless we have $\eta = 1$ i.e. $\eta = \beta$, or 0. Thus the principle of singularity confinement constrains us to these equalities if we wish to detect the integrability candidates. In the case $\alpha \neq 0$ we can always take $\alpha = 1$ and, by translation, put $\beta = 0$. Two cases must be distinguished: $\gamma = 0$ and $\gamma \neq 0$, in which case we can always scale the variables so that $\gamma = -1$. Again a singularity appears whenever $x_n = 0$ in the former case or $x_n = \pm 1$ in the latter. The condition for this singularity not to propagate *ad infinitum* is $\eta = \beta = 0$. (As a matter of fact the essential constraint is $\eta = \beta$. We can always through a translation go back to $\beta \neq 0$, but in this case the translation preserves the equality $\eta = \beta$.) Thus the only instances of (3.7) which have confined singularities are precisely the ones where the mapping belongs to the QRT family $\eta = \beta$ or $\alpha = \eta = 0$. (The full details of this analysis can be found in [1].) The deautonomization of the mappings obtained at this stage consists of assuming that the coefficients are functions of the independent variable n. We ask then what are the constraints on these functions so that the singularity pattern will be exactly the same as in the autonomous case. This results in the following cases, which while nonautonomous do satisfy the singularity confinement criterion, with $z_n = \delta n + z_0$ i.e. linear in the independent variable:

(i) $\alpha = 1, \quad \gamma = -1, \quad \eta = \beta = 0, \quad \epsilon = z_n, \quad \zeta = \text{constant},$
(ii) $\alpha = 1, \quad \gamma = 0, \quad \eta = \beta = 0, \quad \epsilon = z_n, \quad \zeta = \text{constant},$
(iii) $\alpha = 0, \quad \gamma = 0, \quad \eta = \beta = 1, \quad \epsilon = \text{constant}, \quad \zeta = z_n,$
(iv) $\alpha = 0, \quad \gamma = 0, \quad \eta = 0, \quad \beta = 1, \quad \epsilon = \text{constant}, \quad \zeta = z_n.$

Case (i) corresponds to the well known form of d-$\mathbb{P}_{\mathrm{II}}$ while the remaining three are alternate forms of d-$\mathbb{P}_{\mathrm{I}}$. In what follows we shall not present the details concerning the application of singularity confinement for constraining the parameters of the mapping and their deautonomization. The results for the former can be found essentially in [14] while for the latter the full analysis was presented in [18].

One notable exception to integrability for (3.7) is the polynomial mapping

$$x_{n+1} + x_{n-1} = P(x_n) \tag{3.8}$$

where P is a quadratic polynomial, i.e. $\alpha = \beta = 0$. The latter was shown to be nonintegrable by AHH and, by a slightly different approach, by two of the present authors in [16].

The next mapping we shall examine is related to the q-$\mathrm{P_{III}}$ family

$$x_{n+1}x_{n-1} = \frac{P(x_n)}{Q(x_n)}. \tag{3.9}$$

This was one of the mappings examined by Ablowitz *et al* but the fact that they considered coefficients linear in n does not allow one to directly apply their conclusions to q-$\mathrm{P_{III}}$, where the coefficients are exponential in n. Still their main result stands: all the solutions of (3.9) are of infinite order (except a finite number of constant solutions) if the maximum of the degrees of P, Q exceeds 2. The main ingredient in the proof of this result is the inequality (2.8) $T(r; x_{n+1}x_{n-1}) \preceq T(r; x_{n+1}) + T(r; x_{n-1})$, thus we have again (3.2) with $u = 2$ and $v = 0$. The general form of (3.9) with quadratic P, Q is

$$x_{n+1}x_{n-1} = \frac{\eta x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \beta x_n + \gamma}. \tag{3.10}$$

Again, the application of singularity confinement to (3.10) results in the QRT constraint $\eta = \gamma$ or $\eta = \alpha = 0$. The deautonomization of this form, presented in [18], leads to the q-$\mathrm{P_{III}}$ equation as well as mappings which are q-discrete forms of $\mathrm{P_{II}}$ and $\mathrm{P_{I}}$.

Next we turn to the d-$\mathrm{P_{IV}}$ family

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{P(x_n)}{Q(x_n)}. \tag{3.11}$$

First, we apply naïvely (2.8) and (2.9), which gives us, with the notations of (3.2), $u = 2$, $v = 2$ and thus for $w > 4$ we have x_n of infinite order. Thus the only acceptable P, Q can be quartic at the maximum. However, we can produce a more refined estimate using the inequality (2.10). To do this we rewrite (3.11) as

$$x_{n+1}x_{n-1} + x_n x_{n+1} + x_n x_{n-1} = \frac{P(x_n) - x_n^2 Q(x_n)}{Q(x_n)}. \tag{3.12}$$

(Note that since P/Q is irreducible, the rhs of (3.12) is equally irreducible.) Using (2.10) we find $u = 2$, $v = 1$ and so we have $w \leqslant 3$. Thus for integrability candidates we can have for the degree of Q a maximum of $q \leqslant 3$ and for the degree of $P - x^2 Q$ also a maximum at 3. From what we saw above the degree of P is $p \leqslant 4$ and if $q = 3$, then the degree of the numerator would be 5 which is forbidden. Thus we can have at most $q = 2$ and $P = x^2 Q + R$ where R is a polynomial of at most a cubic in x. The well known discrete $\mathrm{P_{IV}}$ falls precisely in this class. As a matter of fact the precise application of singularity confinement to the mapping

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{\alpha x_n^4 + \eta x_n^3 + \kappa x_n^2 + \theta x_n + \mu}{\alpha x_n^2 + \beta x_n + \gamma} \tag{3.13}$$

results in the QRT form: $\eta = \theta = 0$. Further the deautonomization of the mapping yields d-$\mathrm{P_{IV}}$. On the other hand if $q = 1$ then we have $p \leqslant 3$. The mapping then has the form

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{\eta x_n^3 + \kappa x_n^2 + \theta x_n + \mu}{\beta x_n + \gamma}. \tag{3.14}$$

Singularity confinement leads to two distinct subcases. One corresponds to $\alpha = 0$ in (3.13) while the other leads to the constraints $\eta = \beta$ and $\beta\mu = \kappa\theta$ and corresponds to the case where the rhs of (3.13) is not irreducible. In the case $q = 0$ we have three possibilities. Two come from $\eta = \beta = 0$ in (3.14), which entails either $\kappa = 0$ or $\theta = 0$, the latter case being equivalent to $\alpha = \beta = 0$ in (3.13). A third case is *formally* obtained by $\eta = \beta = 0$ and $\kappa = \gamma$ in (3.14) and corresponds to the case where the rhs of (3.14) is not irreducible. So to summarize, the rhs, if polynomial, must be either $ax_n + b$, $ax_n^2 + b$ or $x_n^2 + ax_n + b$. Note that in all three cases, $p \leqslant 2$ rather than $p = 3$. This limit on p when $q = 0$ can in fact be proven by Nevanlinna-type methods. The deautonomizations of these mappings were presented in detail in [18].

The next mapping we are going to examine is the q-$\mathrm{P_V}$ family

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{P(x_n)}{Q(x_n)}. \tag{3.15}$$

The straightforward application of (2.8) and (2.9) gives $u = 2$, $v = 2$. Thus, just as in the case of d-$\mathrm{P_{IV}}$, for $w > 4$ we have a generic x_n of infinite order and the only acceptable P and Q are quartic. However, we can rewrite (3.15) as

$$x_{n+1}x_{n-1} - \frac{x_{n+1}}{x_n} - \frac{x_{n-1}}{x_n} = \frac{P(x_n) - Q(x_n)}{x_n^2 Q(x_n)}. \tag{3.16}$$

Using (2.10) we find $u = 2$, $v = 1$ and thus we must have $w \leqslant 3$. However, this does not mean that the degree of $P - Q$ and $x^2 Q$ must be less or equal to 3 because although P/Q is irreducible, $P - Q$ may have one or more x factors. We are thus led to the examination of each particular case. If $q \leqslant 1$ then no x factorization is necessary and in this case we have $p \leqslant 3$. If $q \leqslant 2$ and one x factorizes, we have $P = Q + xR$ where R is cubic at maximum and $p \leqslant 4$. (Note that even if $q \leqslant 1$, $p = 4$ is allowed because R may still be cubic.) Finally if $q \leqslant 3$, a factor x^2 is necessary and $P = Q + x^2 S$, with S at most quadratic so that $p \leqslant 4$. For $q \leqslant 2$, this case is a subcase of the previous one with $R = xS$. But for $q = 3$, this case can be shown to have its generic solution of infinite order. Indeed one can show that, though the usual method gives (3.2) with $u = 2$, $v = 1$, a more refined calculation leads in this particular case to a situation where $u = 2$, but the 'effective' v is zero, which combined with $w = 3$ leads to a growth of $T(r)$ faster than λ^r with $\lambda = 3/(2 + 2\epsilon)$ for ϵ arbitrarily small when r is large enough. First, let us remark that x does not divide Q (otherwise x would also divide $P = Q + x^2 S$, but P/Q has been assumed to be irreducible). Thus Q has three roots none of which is zero. Since the degree of S is less than that of Q, the affinity of the rhs of (3.16), S/Q, for infinity is entirely due to the affinity of x for each of the three roots of Q i.e. $3T(r; x)$ (up to a bounded correction). In the lhs x_{n+1}, x_{n-1} do contribute to the affinity for infinity as usually $2(1 + \epsilon)T(r + 1; x)$ i.e. $u = 2$. However, since none of the roots of Q is zero, the $1/x_n$ terms do *not* contribute to the affinity for infinity when the rhs is near infinity. Thus it is as if v were zero and only the contribution from x_{n+1}, x_{n-1} balances the contribution $3T(r; x)$ of the rhs.

The case of a quadratic Q corresponds to q-$\mathrm{P_V}$. The general form of the mapping is

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\eta x_n^4 + \theta x_n^3 + \mu x_n^2 + \kappa x_n + \gamma}{\alpha x_n^2 + \beta x_n + \gamma}. \tag{3.17}$$

The application of singularity confinement leads to the constraints $\eta = \gamma$, $\theta = \kappa$, which reduce the mapping to its QRT form. The deautonomization of the latter was presented in detail in [18]. In the case of linear Q we have *a priori*

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\eta x_n^4 + \theta x_n^3 + \mu x_n^2 + \kappa x_n + \lambda}{\beta x_n + \gamma}. \tag{3.18}$$

The constraints resulting from singularity confinement lead to three cases. Either $\eta = \lambda = \gamma$ and $\theta = \kappa$, which comes from taking $\alpha = 0$ in (3.17), or $\eta = 0$, $\lambda = \gamma$ and $\theta(\theta - \kappa) - \gamma(\gamma - \mu) = 0$, which comes from a situation where (3.17) is not irreducible and a nontrivial factor drops out, and finally $\eta = \theta = 0$, $\lambda = \mu$. The latter comes from a trivial simplification by x in (3.17) where $(\eta =)\gamma = 0$, so $\lambda = \mu$ is a consequence of $\theta = \kappa$ in (3.17) and we *do not* require $\lambda = \gamma$ in that case.

Finally we will just list the possible forms of the rhs, when $q = 0$, once the singularity confinement is implemented. Each form comes from (3.17) either by some special values ($\alpha = \beta = 0$, for instance), and/or by trivial (by x) and/or nontrivial simplifications. The possible forms are $x^4 + ax^3 + bx^2 + ax + 1$, $ax^3 + bx^2 + cx + 1$ with $a^2 - ac + b - 1 = 0$, $ax^2 + bx + a$, $ax^2 + bx + 1$, $ax + b$ where any number of the coefficients a, b, c may vanish.

For the remaining discrete Painlevé equations a direct application of the Nevanlinna method is inconvenient. Thus we shall follow a slightly different approach and instead of a single discrete equation we consider the system

$$x_{n+1} * x_n = \frac{P(y_n)}{Q(y_n)} \qquad (3.19a)$$

$$y_n \star y_{n-1} = \frac{R(x_n)}{S(x_n)} \qquad (3.19b)$$

where $*$ and $\star$ stand for either of the operators + or $\times$ (and the use of two different symbols stresses the fact that they can be chosen independently). In order to estimate the order of the solutions of (3.19) we calculate the characteristic of both members of (3.19a) and (3.19b). We have

$$T(r; x_{n+1}) + T(r; x_n) \succeq wT(r; y_n) \qquad (3.20a)$$

$$T(r; y_n) + T(r; y_{n-1}) \succeq \omega T(r; x_n) \qquad (3.20b)$$

where w, ω are the maxima of the degrees of P, Q and R, S, respectively. Next we compute (3.20a) once downshifted, i.e. at the point $(n - 1)$, and eliminating $T(r; y)$ we find

$$T(r; x_{n+1}) + 2T(r; x_n) + T(r; x_{n-1}) \succeq w\omega T(r; x_n). \qquad (3.21)$$

Using (2.12) we have

$$2(1 + \epsilon)T(r + 1; x_n) \succeq (w\omega - 2)T(r; x_n) \qquad (3.22)$$

which means that (apart from a finite number of constant solutions) the generic x is of infinite order if $w\omega > 4$. Given this constraint integrability candidates may only have $w\omega \leqslant 4$. We must thus examine the cases of $w = \omega = 2$ and $w = 1$, $\omega \leqslant 4$ (or the equivalent one $\omega = 1$, $w \leqslant 4$).

The case $w = \omega = 2$ leads (after the singularity confinement constraints have been implemented) to well known integrable equations. In all cases, we present only the generic equation in which we assume that both w and ω are exactly 2. The cases where one (or both) rhss are homographic are treated later. Still, various subcases do exist, coming from special values of the parameters. The purely multiplicative case is

$$x_{n+1}x_n = \frac{\kappa y_n^2 + \lambda y_n + \mu}{\alpha y_n^2 + \beta y_n + \gamma} \qquad y_n y_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa}. \qquad (3.23)$$

When deautonomized using singularity confinement, this becomes the ‘asymmetric’ q-P_{III}, discrete P_{VI}, obtained by Jimbo and Sakai [19] or some degenerate form thereof.

The purely additive form is the ‘asymmetric’ d-P_{II}, discrete P_{III}, we have studied in [20]

$$x_{n+1} + x_n = \frac{\delta y_n^2 + \epsilon y_n + \zeta}{\alpha y_n^2 + \beta y_n + \gamma} \qquad y_n + y_{n-1} = \frac{\beta x_n^2 + \epsilon x_n + \lambda}{\alpha x_n^2 + \delta x_n + \kappa}. \qquad (3.24)$$

Finally the mixed case is a discrete P_V [21]

$$x_{n+1} + x_n = \frac{\delta y_n^2 + \epsilon y_n + \zeta}{\alpha y_n^2 + \beta y_n + \gamma} \qquad y_n y_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa}. \tag{3.25}$$

Let us now turn to the case where $\omega \leqslant 4$, $w = 1$. The latter means that the rhs of (3.19a) is just homographic in y. Solving for y we obtain $y_n = H(x_{n+1} * x_n)$ where H is homographic in its argument. We obtain thus the following mapping:

$$H(x_{n+1} * x_n) \star H(x_n * x_{n-1}) = \frac{R(x_n)}{S(x_n)} \tag{3.26}$$

where the degrees of R and S are not larger than 4. Four cases must be distinguished depending on the choices of $*$ and $\star$. When $*$ stands for $+$, the resulting equation is the autonomous form of a discrete P_V (once the singularity confinement constraints have been implemented in the most generic case, where R/S is irreducible with both R and S quartic). When $*$ stands for $\times$ the final equation is the autonomous form of a q-P_{VI}. If $\star$ is taken as $\times$, the two forms are the standard ones as obtained in [22]:

$$\frac{(x_{n+1} + x_n - 2\zeta)}{(x_{n+1} + x_n)} \frac{(x_{n-1} + x_n - 2\zeta)}{(x_{n-1} + x_n)} = \frac{\alpha(x_n - \zeta)^4 + \beta(x_n - \zeta)^2 + \gamma}{\alpha x_n^4 + \delta x_n^2 + \epsilon} \tag{3.27}$$

$$\frac{(x_{n+1}x_n - \zeta^2)}{(x_{n+1}x_n - 1)} \frac{(x_{n-1}x_n - \zeta^2)}{(x_{n-1}x_n - 1)} = \frac{\alpha(x_n^4 + \zeta^4) + \beta x_n(x_n^2 + \zeta^2) + \gamma x_n^2}{\alpha(x_n^4 + 1)\delta x_n(x_n^2 + 1) + \epsilon x_n^2}. \tag{3.28}$$

If $\star$ is the operator $+$ no new equations result. It is straightforward to show that by adding the adequate constant to both sides of the equation, the resulting forms are just the ones obtained in (3.27), (3.28). At this stage, ζ is a constant, like all the other parameters. The deautonomization was presented in [22]. We will not comment on the hosts of special forms R and S can assume through special values of the coefficients and various factorizations.

4. Conclusion

In this paper we have introduced a novel integrability criterion for discrete systems. It combines the two features which are presumably important for integrability: growth and singularities. The growth part is taken care of by the study of the Nevanlinna characteristic. Our approach consists of focusing on autonomous systems for the study of the behaviour at infinity of the solution. The rationale behind this approach is the following. We believe that for autonomous systems, for which the growth of solutions is not marred by too-fast growing coefficients, integrability is to be associated to finite order. Once systems with infinite-order solutions are rejected at this step one can proceed safely to deautonomization through the singularity confinement criterion. In a sense the finite-order requirement was the ingredient which was missing in order to turn singularity confinement into a powerful integrability detector.

The application of this method, which was dubbed the three-tiered approach, was carried out in detail for mappings of the QRT family, which include the discrete $\mathbb{P}$s. Our choice of these systems was dictated by the fact that they are among the most well studied. Moreover for a large number of d-$\mathbb{P}$s integrability is well established through the existence of a Lax pair. For the remaining equations, although this is certainly not a proof, there exists an independent, strong indication of integrability obtained through algebraic entropy–low-growth techniques [23]. For most of the discrete Painlevé equations we have presented a geometrical description [24] (based on affine Weyl groups) which makes possible the construction of their solutions starting from those of nonautonomous Hirota–Miwa equations, which constitutes (at least in the eyes of the present authors) a further indication of integrability.

This paper strengthens further the results obtained on the integrability of discrete Painlevé equations. Moreover it shows that, at least within the QRT parametrization, the known d-$\mathbb{P}$s are the only ones one can expect to be integrable. (A *caveat* is unavoidable here concerning the special limits and degenerate forms of equations belonging to the family of d-$\mathrm{P_V}$ and q-$\mathrm{P_{VI}}$ which have not yet been exhaustively studied.) This conclusion should not be interpreted as indicating that studies on the possible forms of d-$\mathbb{P}$s have been exhausted. There remains a whole *terra incognita* of two variable discrete systems which cannot be written as (3.19) [25] which have never been the object of a systematic investigation. Moreover the domain of higher d-$\mathbb{P}$s has barely been touched upon. We expect the methods developed in this paper to provide valuable tools for these explorations.

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